

## Graphene and the Zermelo Optical Metric of the BTZ Black Hole

M. Cvetič<sup>1,2</sup> and G.W Gibbons<sup>3</sup><sup>1</sup>*Department of Physics and Astronomy, University of Pennsylvania, Philadelphia, PA 19104-6396, USA*<sup>2</sup>*Center for Applied Mathematics and Theoretical Physics, University of Maribor, Maribor, Slovenia*<sup>3</sup>*D.A.M.T.P., University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, U.K.*

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**Abstract**

We argue that the low energy electron excitations of the curved graphene sheet  $\Sigma$  are solutions of the massless Dirac equation on a  $2+1$  dimensional ultra-static metric on  $\mathbb{R} \times \Sigma$ . An externally applied magnetic field on the graphene sheet induces a gauge potential on the world volume of the membrane which could be mimicked by considering a stationary optical metric of the Zermelo form, which is conformal to the BTZ black hole when the sheet has a constant negative curvature. We show that there is fundamental geometric obstacle to obtain a model that extends all the way to the black hole horizon.

# 1 Introduction

In a recent paper [1] it has been suggested that a Beltrami surface of revolution with constant negative curvature (Beltrami Trumpet) made from graphene may exhibit some of the interesting effects which arise in quantum field theory in a curved spacetimes.

The idea of [1] is that on a curved graphene sheet  $\Sigma$  there are low energy electronic excitations which satisfy the massless Dirac equation on a  $2 + 1$  dimensional ultra-static<sup>1</sup> metric on  $\mathbb{R} \times \Sigma$ . In [1] it is assumed that the physical graphene metric is of ultra static form

$$ds^2 = -dt^2 + h_{ij}dx^i dx^j, \quad i = 1, 2 \quad (1)$$

and that  $\Sigma, h_{ij}$  may be isometrically embedded as a surface of revolution in Euclidean three space  $\mathbb{E}^3$  with coordinates  $(x(x^i), y(x^i), z(x^i))$ . Since in the case of graphene, there is no obvious source of red shifting, the assumption made in [1] that  $g_{tt} = -1$  appears to be physically well justified. The massless Dirac equation is invariant under conformal rescalings, that is, in  $D$  spacetime dimensions, if

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \tilde{\Psi} = \frac{1}{\Omega^{\frac{D-1}{2}}} \Psi \quad (2)$$

we have

$$\gamma^\mu \nabla_\mu \Psi = \frac{1}{\Omega^{\frac{D+1}{2}}} \tilde{\gamma}^\mu \tilde{\nabla}_\mu \Psi. \quad (3)$$

Moreover, any static metric is locally conformally ultra static, that is

$$-V^2 dt^2 + g_{ij} dx^i dx^j = V^2 \left\{ -dt^2 + h_{ij} dx^i dx^j \right\}, \quad (4)$$

where  $h_{ij} = V^{-2} g_{ij}$  is called [2] the *optical metric* of the static metric on the left hand side of (4), *classically* at least there might appear to be no obstacle to considering ultra-static metrics for graphene of the form (1). However it is possible to consider graphene, or indeed any other two-dimensional film, subject to an externally applied magnetic field. This would induce a gauge field  $A_\mu$  on the “world volume” of the membrane or 2-brane. From a gravitational point of view such a gauge connection, which would break time reversal invariance but not conformal invariance, could be mimicked by considering a *stationary* rather than a static metric. Locally any stationary metric may be brought by a conformal rescaling [3] to the *Zermelo* form

$$ds^2 = -dt^2 + h_{ij}(dx^i - W^i dt)(dx^j - W^j dt) \quad (5)$$

where  $h_{ij}$  is called the Zermelo optical metric and  $W^i$  the *wind vector field*. We shall show explicitly later, that, at least in the case of a axial symmetry, the effect of the wind is equivalent to an induced magnetic connection  $A_\mu$ . Thus, as we shall show in detail later in the paper, we could in principle construct a graphene analogue of the *conformal geometry* of a rotating two-dimensional black hole such as that of BTZ [4]. Actually as we shall show later, there is a Fundamental geometric obstacle, already encountered in [1] to obtaining a model system which extends all the way to the horizon.

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<sup>1</sup>A locally static metric is one which is time independent, time reversal invariant. A globally static metric is one for which  $g_{tt}$  is non-zero throughout the entire spacetime manifold. A globally static metric cannot therefore contain a Killing horizon (unless it is Killing horizon of some other Killing vector field). An ultra static metric is one for which the gravitational red shifting does not occur, i.e one for which  $g_{tt}$  is independent of the spatial coordinates. As a consequence a massive particle may remain at rest in such a spacetime because it suffers no gravitational attraction. An ultra-static metric is thus necessarily globally static and cannot contain a Killing horizon.

## 2 Surfaces of revolution

From now on we shall confine attention to axisymmetric metrics which may be isometrically embedded in Euclidean space  $\mathbb{E}^3$  as surfaces of revolution. We have

$$h_{ij}dx^i dx^j = d\rho^2 + C^2(\rho)d\phi^2, \quad 0 \leq \phi < 2\pi, \quad (6)$$

so that

$$C^2(\rho) = x^2 + y^2 = r^2 \quad (7)$$

and

$$d\rho^2 = dr^2 + dz^2. \quad (8)$$

The Gauss curvature is given by

$$K = -\frac{C''}{C}. \quad (9)$$

### 2.1 The Beltrami Trumpet

The example introduced in [1] is given by the *Beltrami's Trumpet*

$$C(\rho) = a \exp\left(-\frac{\rho}{a}\right), \quad \rho \geq 0, \quad \Rightarrow \quad K = -\frac{1}{a^2}. \quad (10)$$

If

$$w = a\phi + ia \exp\left(\frac{\rho}{a}\right), \quad (11)$$

then

$$h_{ij}dx^i dx^j = \frac{a^2 |dw|^2}{(\Im w)^2}, \quad (12)$$

which would be the standard model of  $H^2$  as the upper half complex plane, if  $\Im w \geq 0$  but we must quotient by the  $\mathbb{Z}$  action  $w \rightarrow w + 2\pi a$  and moreover take  $\Im w \geq a$ . As is well known, the Beltrami Trumpet is obtained by revolving a *tractrix* curve about the  $z$ -axis. As one may verify, the tractrix is the locus of one end of a light rope of length  $a$  to which is attached to a heavy weight as the other end is dragged along the negative  $z$  axis. Initially the rope occupies the interval  $0 \leq x \leq a, y = 0, z = 0$  and there is a half cusp at  $(a, 0, 0)$ .

This failure to embed all of  $H^2/\mathbb{Z}$  is not an artifact of our symmetry assumptions. A theorem of Hilbert implies that we cannot get a non-singular embedding of any complete metric of constant negative curvature into three dimensional Euclidean space. As we see shortly this in turn implies that no non-singular embedding of the optical metric of a two-dimensional black hole can ever reach the horizon. It must always terminate at a finite proper distance from it.

### 2.2 Fullerenes

At the microscopic level, this idea has already been applied to Fullerenes [6]. These are approximately spherical and so the appropriate spacetime is  $\mathbb{R} \times S^2$ , the  $2+1$  dimensional version of Einstein's Static Universe. In fact like  $\mathbb{R} \times H^2$ ,  $\mathbb{R} \times S^2$  is also conformally flat. However this plays no direct role in the analysis of the Fullerene spectrum, and there is no apparent evidence for any thermal effects. In fact the spin-connection on  $S^2$  is interpreted as a magnetic monopole located at the centre of the Fullerene molecule.

It is also possible to insert insert conical defects on  $S^2$  and retain both axisymmetry and embedability. If

$$x = k \sin \theta \cos \frac{\phi}{k} \quad (13)$$

$$y = k \sin \theta \sin \frac{\phi}{k} \quad (14)$$

$$z = E(k, \theta), \quad (15)$$

with

$$E(k, \theta) = \int_{\theta}^0 \sqrt{1 - k^2 \cos^2 \theta'} d\theta', \quad (16)$$

we get an isometric immersion of  $S^2$  into  $\mathbb{E}^3$  which if  $k \neq 1$  is branched over the north and south poles [7].

The induced metric is

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2 \quad (17)$$

which is of constant Gauss curvature, but if one takes  $0 \leq \phi \leq 2\pi k$ , there will be a conical singularity at the north and south poles. The resulting surface will thus have the shape of an American football.

Actually this shape has been used as an internal space in six dimensional models. In that case the distributional sources at the north and south poles are interpreted as three branes. Perhaps an alternative interpretation is as the branched covering of a 5-brane wrapped on an  $S^2$ .

### 2.3 Near Horizon Geometry of a static BTZ Black Hole

For vanishing angular momentum  $J$  the BTZ metric is

$$ds_{BTZ}^2 = -(\frac{\tilde{r}^2}{l^2} - M)dt^2 + \frac{d\tilde{r}^2}{(\frac{\tilde{r}^2}{l^2} - M)} + \tilde{r}^2 d\phi^2, \quad 0 \leq \phi < 2\pi. \quad (18)$$

The associated *optical metric* is

$$ds_o^2 = -dt^2 + \frac{d\tilde{r}^2}{(\frac{\tilde{r}^2}{l^2} - M)^2} + \frac{\tilde{r}^2}{(\frac{\tilde{r}^2}{l^2} - M)} d\phi^2, \quad \Omega^2 = (\frac{\tilde{r}^2}{l^2} - M). \quad (19)$$

We set

$$\tilde{r}_+ = l\sqrt{M}, \quad a = 2l, \quad (\tilde{r} - \tilde{r}_+) = \frac{1}{8}\tilde{r}_+ \exp(-2\frac{\rho}{a}), \quad (20)$$

and expand about the horizon to see that

*To lowest order the region  $\tilde{r} \geq \frac{9}{8}\tilde{r}_+$  of the optical metric of BTZ black hole maps onto the Beltrami Trumpet spacetime.*

## 3 Exact Non-rotating BTZ black hole

We will embed into  $\mathbb{E}^3$  the BTZ optical geometry. Thus

$$\frac{1}{l^2} ds_o^2 = -dt^2 + l^2 \frac{dr^2}{(r^2 - a^2)^2} + \frac{r^2}{r^2 - a^2} d\phi^2, \quad (21)$$

$$C^2 = \frac{r^2}{r^2 - a^2}, \quad (22)$$

and

$$dz^2 + dC^2 = \frac{l^2 dr^2}{(r^2 - a^2)^2}, \quad \Rightarrow \quad dz^2 = l^2 \frac{(r^2 - a^2(1 + \frac{a^2}{l^2})) dr^2}{(r^2 - a^2)^3}. \quad (23)$$

Thus

$$\left(\frac{dz}{dC}\right)^2 = \frac{1 + \frac{l^2}{a^2} - C^2}{C^2 - 1}. \quad (24)$$

Clearly the embedding must stop at the radius for which  $C = \sqrt{1 + \frac{l^2}{a^2}}$ . This is outside the horizon for which  $C^2 \rightarrow \infty$ . The radial optical distance  $\rho$  is given by

$$d\rho = l \frac{dr}{r^2 - a^2}, \quad (25)$$

Thus

$$\frac{r}{a} = \coth\left(\frac{a}{l}\rho\right), \quad C = \cosh\left(\frac{a}{l}\rho\right), \quad (26)$$

and so the optical metric is

$$\frac{1}{l^2} ds_o^2 = -dt^2 + d\rho^2 + \cosh^2\left(\frac{a}{l}\rho\right) d\phi^2, \quad (27)$$

and the BTZ metric itself is

$$ds_{BTZ}^2 = \frac{a^2}{\sinh^2\left(\frac{a}{l}\rho\right)} \left\{ -dt^2 + d\rho^2 + \cosh^2\left(\frac{a}{l}\rho\right) d\phi^2 \right\}. \quad (28)$$

*Note that the Gaussian curvature of the spatial part of the BTZ optical metric is of constant negative curvature.*

Assuming that such sheets can be made in the Laboratory such BTZ trumpets could also be made.

*Note also that the BTZ optical geometry is locally of the form  $\mathbb{R} \times H^2$  and hence conformally flat.*

### 3.1 Zermelo Interpretation of the BTZ black hole

This follows [3]. If

$$\Delta(r) = \frac{r^2}{l^2} - M + \frac{J^2}{4r^2} \quad (29)$$

then the BTZ metric with non-vanishing angular momentum  $J$  takes the form

$$ds_{BTZ}^2 = \Delta \left\{ -dt^2 + \frac{dr^2}{\Delta^2} + \frac{r^2}{\Delta} \left( d\phi - \frac{J}{2r^2} dt \right)^2 \right\}. \quad (30)$$

The metric in the braces is of Zermelo form (5) with

$$h_{ij} dx^i dx^j = \frac{dr^2}{\Delta^2} + \frac{r^2}{\Delta} d\phi^2, \quad (31)$$

$$W^i \partial_i = \frac{J}{2r^2} \partial_\phi. \quad (32)$$

In the near horizon limit described above  $h_{ij}$  is of Beltrami trumpet form.

## 4 Gauge Fields

External magnetic fields applied to a graphene surface of revolution will induce a non trivial gauge  $U(1)$  field. Suppose that the gauge field is given by

$$A_\mu dx^\mu = A(C)d\phi, \quad \Rightarrow \quad F = \frac{A'}{C} C dC \wedge d\phi = \frac{A'}{C} dx \wedge dy. \quad (33)$$

In the embedding space this will correspond to a magnetic field

$$B_z = \frac{A'}{C} \quad (34)$$

Thus an applied uniform field of strength  $B_0$  has  $A = B_0 C$ . On the surface however since

$$F = \frac{A'(C)}{C} \frac{dC}{d\rho} d\rho \wedge C d\phi = \frac{A'(C)}{C} \frac{dC}{d\rho} \eta, \quad (35)$$

where  $\eta = d\rho \wedge C d\phi$  is the area 2-form on the surface, the normal field strength  $B_n$  is

$$B_n = \frac{A'(C)}{C} \frac{dC}{d\rho}. \quad (36)$$

Thus a constant field  $B_0$  will produce a normal field

$$B_n = B_0 \frac{1}{C} \frac{dC}{d\rho} = B_0 \frac{a}{l} \tanh\left(\frac{a}{l} \rho\right) \quad (37)$$

In order to produce a uniform field on the surface we need to set

$$\frac{A'(C)}{C} \frac{dC}{d\rho} = B_c = \text{constant}. \quad (38)$$

Thus

$$\frac{dA}{d\rho} = B_c C. \quad (39)$$

That is

$$B_z = B_c \left(\frac{dC}{d\rho}\right)^{-1} = \frac{l B_c}{a \sinh\left(\frac{a}{l} \rho\right)}. \quad (40)$$

Note that the angle  $\psi$  that the meridians make with the vertical direction is given by

$$\sin \psi = \frac{dC}{d\rho}, \quad (41)$$

and so these formulae are geometrically rather obvious. in particular we have

$$B_n = B_z \sin \psi \quad (42)$$

## 5 The Dirac equation

### 5.1 The static case

We have

$$(\gamma^i \nabla_i + \gamma^0 \partial_t) \Psi = 0, \quad (43)$$

where  $(i, j)$  are dyad indices and  $de^i = -\omega^i{}_j \wedge e^j$

$$\nabla \Psi = d\psi + \frac{1}{4}\omega_{ij}\gamma^i\gamma^j\Psi. \quad (44)$$

For a surface of revolution

$$e^1 = d\rho, \quad e^2 = C(\rho)d\phi, \quad (45)$$

$$de^2 = \frac{C'}{C}e^1 \wedge e^2, \quad \Rightarrow \quad \omega_{21} = \frac{C'}{C}e^2. \quad (46)$$

Thus

$$\nabla = d + \gamma^2\gamma^1\frac{C'}{2C}e^2, \quad (47)$$

and

$$(\gamma^1\partial_\rho + \gamma^2\frac{1}{C}\partial_\phi + \gamma^2\gamma^1\frac{C'}{2C} + \gamma^0\partial_t)\Psi = 0. \quad (48)$$

That is

$$\gamma^1\frac{1}{\sqrt{C}}\partial_\rho(\sqrt{C}\Psi) + \gamma^2\frac{1}{C}\partial_\phi\Psi + \gamma^0\partial_t\Psi = 0, \quad (49)$$

If we set

$$\sqrt{C}\Psi = \tilde{\Psi}, \quad \tilde{\gamma}^i = \gamma^0\gamma^i \quad (50)$$

we obtain ( we use  $-++$  signature)

$$\tilde{\gamma}^1\partial_\rho\tilde{\Psi} + \tilde{\gamma}^2\frac{1}{C}\partial_\phi\tilde{\Psi} - \partial_t\tilde{\Psi} = 0. \quad (51)$$

Now both  $\gamma^3$  and  $\gamma^0\tilde{\gamma}^1\tilde{\gamma}^2$  commute with (51) and themselves, and have eigenvalues  $\pm$ .

## 5.2 The Stationary Case

We consider the metric:

$$-dt^2 + \rho^2 + C^2(\rho)(d\phi - W(\rho)dt)^2, \quad (52)$$

and define a pseudo-orthonormal basis of one-forms by

$$e^0 = dt, \quad e^1 = d\rho, \quad e^2 = C(d\phi - Wdt), \quad (53)$$

and a dual basis of vector fields by

$$e_0 = \partial_t + W\partial_\phi, \quad e_1 = \partial_\rho, \quad e_2 = \frac{1}{C}\partial_\phi. \quad (54)$$

Note that  $e_a\phi = \delta_a^0 W(\rho)$  and so the dreibein  $e_a$  is differentially rotating. Using the formulae

$$de^a = -\omega^a{}_b \wedge e^b, \quad \omega_{ab} = \eta_{ac}\omega^c{}_b = -\omega_{ba}, \quad (55)$$

with  $a = 0, 1, 2$  and  $\eta_{ab} = \text{diag}(-1, 1, 1)$  we find

$$\omega_{01} = -\frac{1}{2}CW'e^2, \quad \omega_{02} = -\frac{1}{2}CW'e^1, \quad \omega_{21} = -\frac{1}{2}CW'e^0 + \frac{C'}{C}e^2. \quad (56)$$

The massless Dirac equation may be written as

$$\left(\gamma^a e_a + \frac{1}{4}\gamma^a\omega_{abc}\gamma^b\gamma^c\right)\Psi = 0, \quad (57)$$

where  $\omega_{abc}$  are what are sometimes called *Ricci rotation coefficients*

$$\omega_{bc} = e^a \omega_{abc}. \quad (58)$$

Thus

$$\left( \gamma^1 (\partial_\rho + \frac{1}{2} \frac{C'}{C}) + \gamma^2 \frac{1}{C} \partial_\phi + \gamma^0 (\partial_t + W \partial_\phi) + \frac{1}{4} \gamma^0 \gamma^1 \gamma^2 C W' \right) \Psi = 0. \quad (59)$$

If we chose

$$\gamma^0 = i\sigma_2, \quad \gamma^1 = \sigma_1, \quad \gamma^2 = \sigma_3. \quad (60)$$

then  $\gamma^0 \gamma^1 \gamma^2 = 1$  and we get a position dependent “mass-like” term and a connection term. If  $\Psi \propto e^{-i\omega t + im\phi}$  we have that

$$-ieA_0 = imW, \quad \Rightarrow eA_0 = -mW, \quad (61)$$

and the effective *electric* field in what is actually a rotating frame is

$$eF = edA = W' d\rho \wedge dt = W' e^1 \wedge e^0. \quad (62)$$

## 6 Conclusion

We argue that the curved graphene sheet with negative constant curvature in the externally applied magnetic field could be modeled by considering a stationary optical metric of the Zermelo form which is conformal to the BTZ black hole. In particular, the magnetic field corresponds to the *wind* of the optical Zermelo metric. Furthermore, we establish that there is a fundamental geometric obstruction to obtain a model that extends all the way to the BTZ black hole horizon. We model the low energy electron excitations of such graphene sheets by studying solutions of the 2 + 1-dimensional massless Dirac equation in the stationary optical metric, conformal to the BTZ black hole.

Related analyses can be applied to any other system described with a two-dimensional curved surface  $\Sigma$ . We should also point out that there are other possible embeddings of a surface with constant negative curvature which are not a surface of revolution but a twisted Beltrami trumpet. It embeds more of  $H^2$  than the Beltrami trumpet. It would be interesting to further study implications of such more general embeddings.

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## References

- [1] Alfredo Iorio, Gaetano Lambiase and Maria A. H. Vozmediano, The Hawking-Unruh phenomenon on graphene, arXiv:1108.2340[cond-mat.mtrl-sci] [v1].<sup>2</sup>
- [2] G. W. Gibbons and C. M. Warnick, ‘Universal properties of the near-horizon optical geometry, Phys. Rev. D **79** (2009) 064031 [arXiv:0809.1571 [gr-qc]].

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<sup>2</sup> Note that there the third author is on [v1], only and that [v1] and [v2] differ considerably.



- [3] G. W. Gibbons, C. A. R. Herdeiro, C. M. Warnick and M. C. Werner, Stationary Metrics and Optical Zermelo-Randers-Finsler Geometry, *Phys. Rev. D* **79** (2009) 044022 [arXiv:0811.2877 [gr-qc]].
- [4] M. Banados, C. Teitelboim and J. Zanelli, The Black hole in three-dimensional space-time, *Phys. Rev. Lett.* **69**, 1849 (1992) [arXiv:hep-th/9204099].
- [5] A. Dasgupta, Emission of fermions from BTZ black holes *Phys Lett B***445**(1999) 279-286 [hep-th/9808086 ]
- [6] J. González, F. Guinea and M. Angeles Vozmediano *Phys. Rev. Lett.* **69** (1992) 172-175
- [7] A Cayley, On the flexure of a spherical surface *Messenger of Mathematics* **7** (1887) 88